Absolute Extrema

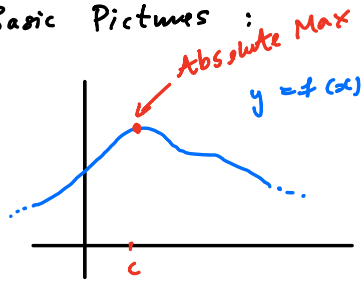
Definition

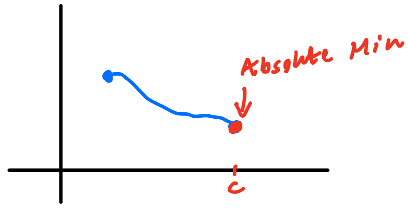
Called absolute extrema

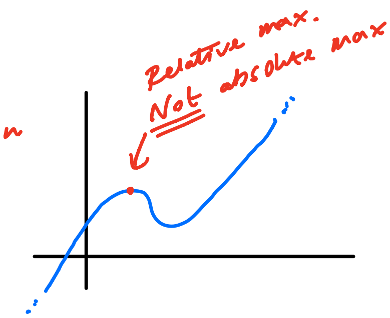
$f(c)$ absolute max : $f(x) \le f(c)$ for all x in domain of f

$f(c)$ absolute min : $f(x) \ge f(c)$ for all x in domain of f

Basic Pictures :





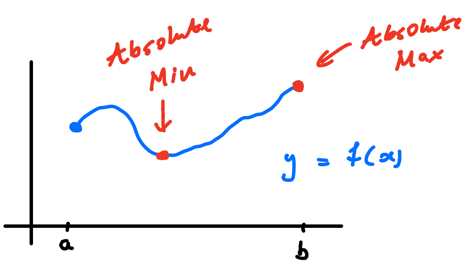


Remark :

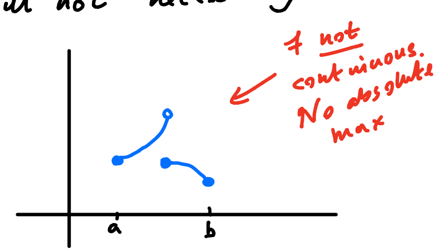
$f(c)$ absolute max $\Rightarrow$ $f(c)$ relative max $\Rightarrow$ c critical number

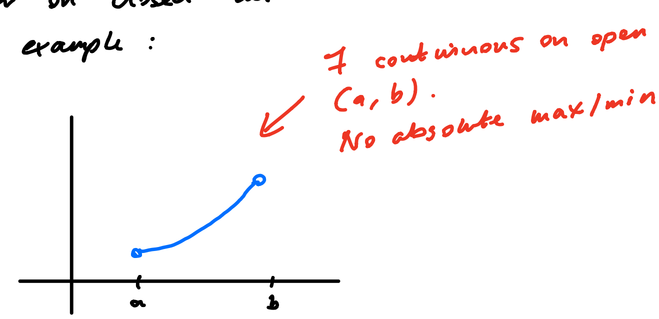
Extreme Value Theorem

f – continuous function on closed interval $[a,b]$. Then f has both an absolute max and an absolute min.



Remark If f not continuous or on closed interval the result will not necessarily be true. For example :





Finding Absolute Max / Min on Closed Interval

(f continuous on $[a,b]$)

1/ Calculate $f'(x)$.

2/ Find all critical numbers : A/ $f'(c)=0$ or B/ $f'(c)$ DNE (remember endpoints are critical)

3/ Evaluate f at all critical numbers

4/ Largest value is absolute max. Smallest value is absolute min.

Example $f(x) = x^{8/3} - 16x^{2/3}$ on $[-1, 8]$

1/ $f'(x) = \frac{8}{3}x^{5/3} - \frac{32}{3}x^{-\frac{1}{3}} = \frac{8}{3}x^{5/3} - \frac{32}{3} \cdot \frac{1}{x^{1/3}}$

2/ A/ $f'(x) = 0 \Rightarrow \frac{8}{3}x^{5/3} - \frac{32}{3}\frac{1}{x^{1/3}} = 0$

$$\Rightarrow \frac{8}{3}x^{5/3} = \frac{32}{3}\frac{1}{x^{1/3}}$$

$$\Rightarrow x^{5/3} = 4 \cdot \frac{1}{x^{1/3}}$$

$$\Rightarrow x^{5/3} \cdot x^{1/3} = 4$$

$$\Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

B/ f' undefined $\Rightarrow x^{1/3} = 0 \Rightarrow x = 0$

endpoints

$\Rightarrow$ on $[-1, 8]$ critical numbers are $-1, 0, 2, 8$

3/ $f(-1) = -15$

$f(0) = 0$

$f(2) = -19.05$

$f(8) = 192$

4/ $f(2)$ absolute min and $f(8)$ absolute max

Q/ : Can we find absolute max/min if f is <u>not</u> on a closed interval?

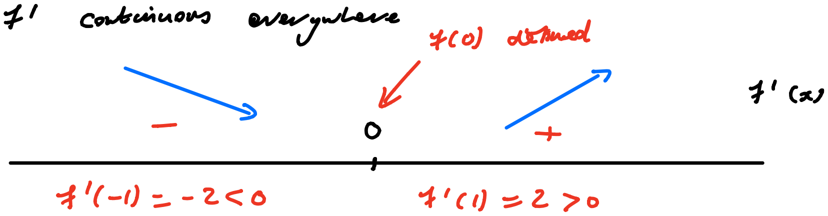
Example $f(x) = x^2$ on $(-\infty, \infty)$

Lets do the usual sign analysis to find relative extrema.

$f'(x) = 2x$

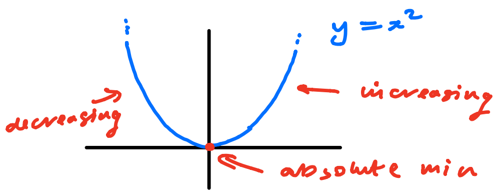
A/ $f'(x) = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$

B/ f' continuous everywhere



$\Rightarrow f(0)$ rel. min.

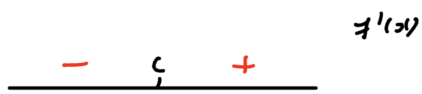
Notice though that f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$ $f(0)$ must be an absolute min



Critical Point Theorem

f – continuous function on interval. Assume there is a single critical number in I (excluding endpoints). Then

$\Rightarrow f(c)$ Absolute Max



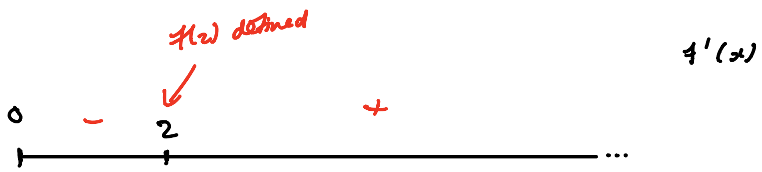
$\Rightarrow f(c)$ Absolute Min

Example Find absolute max / min of $f(x) = 3x^4 - 4x^3 - 12x^2 + 2$ on $[0, \infty)$.

$$f'(x) = 12x^3 - 12x^2 - 24x = 12(x^3 - x^2 - 2x)$$
$$= 12x(x^2 - x - 2) = 12x(x+1)(x-2)$$

A/ $f'(x) = 0 \Rightarrow 12x(x+1)(x-2) = 0 \Rightarrow x = 0, -1, 2$

B/ f' continuous everywhere on $[0, \infty)$



$f'(3) = 12\cdot 3\cdot 4\cdot 1 > 0$

$f'(1) = 12\cdot 1\cdot 2\cdot(-1) < 0$

$\Rightarrow$ $f(2)$ absolute min on $[0,\infty)$

Example Find absolute max/min of $f(x) = \dfrac{\ln(x)}{x^3}$.

Domain $(0,\infty)$

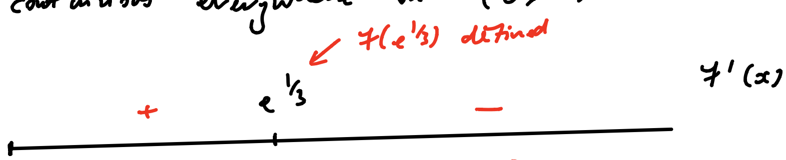
$$f(x) = \frac{u(x)}{v(x)}, \quad u(x) = \ln(x), \quad v(x) = x^3 \quad \Rightarrow \quad \begin{aligned} u'(x) &= \frac{1}{x} \\ v'(x) &= 3x^2 \end{aligned}$$

$$\Rightarrow \quad f'(x) = \frac{u'(x)\,v(x) - u(x)\,v'(x)}{(v(x))^2} = \frac{\frac{1}{x}\cdot x^3 - \ln(x)\cdot 3x^2}{(x^3)^2}$$

$$= \frac{x^2(1 - 3\ln(x))}{x^6} = \frac{1 - 3\ln(x)}{x^4}$$

A/ $f'(x) = 0 \Rightarrow 1 - 3\ln(x) = 0 \Rightarrow \ln(x) = \frac{1}{3} \Rightarrow x = e^{1/3}$

B/ f' continuous everywhere on $(0,\infty)$



$$f'(e) = \frac{1 - 3\ln(e)}{e^4} = \frac{-2}{e^4} < 0$$

$$f'(e^{1/6}) = \frac{1 - 3\ln(e^{1/6})}{e^{4/6}} = \frac{1 - \frac{3}{6}}{e^{4/6}} = \frac{\frac{1}{2}}{e^{4/6}} > 0$$

$\Rightarrow$ $f(e^{1/3})$ absolute max.

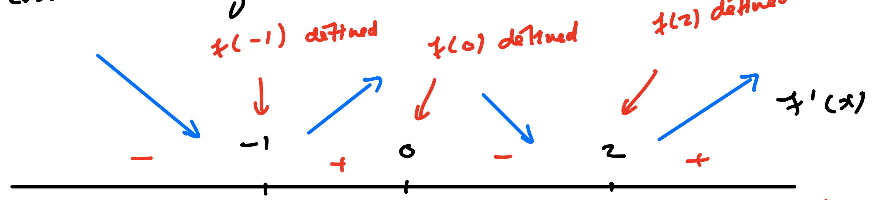
Remark One can also potentially use sign analysis to find absolute max/min when there is more than one critical number.

Example Find absolute max/min of $f(x) = 3x^4 - 4x^3 - 12x^2 + 2$ on $(-\infty,\infty)$.

$$f'(x) = 12x^3 - 12x^2 - 24x = 12(x^3 - x^2 - 2x)$$
$$= 12x(x^2 - x - 2) = 12x(x+1)(x-2)$$

A/ $f'(x) = 0 \Rightarrow 12x(x+1)(x-2) = 0 \Rightarrow x = 0, -1, 2$

B/ f' continuous everywhere on $[0, \infty)$



$f'(-\frac{1}{2}) = 12 \cdot -\frac{1}{2} \cdot \frac{1}{2} \cdot -\frac{3}{2} > 0$ $\quad f'(3) = 12 \cdot 3 \cdot 4 \cdot 1 > 0$

$f'(-2) = 12 \cdot -2 \cdot -1 \cdot -4$ $\quad f'(1) = 12 \cdot 1 \cdot 2 \cdot -1 < 0$

$\Rightarrow$ $f(-1)$, $f(2)$ relative min

$f(0)$ relative max

f decreasing on $(-\infty, -1)$ and f increasing on $(2, \infty)$ $\Rightarrow$ Either $f(-1)$ or $f(2)$ an absolute min

$f(-1) = -3$
$f(2) = -30$ $\Rightarrow$ $f(2)$ absolute min of f on $(-\infty, \infty)$

We cannot determine if there is an absolute max or not based on this analysis

Example